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AG II - Schemes and sheaves

Solutions – week 2

Exercise 1. Let $\iota_i : U_i \to X$ for the inclusion of the open set.
Some quick remarks : using cocycle condition, we get $\varphi_{ii} = \varphi_{ii} \circ \varphi_{ii}$. By hypothesis φ_{ii} are isomorphisms so we get : $\varphi_{ii} = \mathrm{id}_{\mathcal{F}_i}$. Then using the cocycle condition : $\mathrm{id}_{\mathcal{F}_i} = \varphi_{ji} \circ \varphi_{ij}$ and $\mathrm{id}_{\mathcal{F}_j} = \varphi_{ij} \circ \varphi_{ji}$.
We define [1] $\mathcal{F}$ on an open set U by :

$$\mathcal{F}(U) = \{(s_i) \in \prod_i \mathcal{F}_i(U \cap U_i) \mid \forall (i,j) \quad s_{j|_{U_{ij}}} = \varphi_{ij}(s_{i|_{U_{ij}}})\}$$

as *sub-(pre)sheaf of the product sheaf* $\prod_i \iota_{i*}\mathcal{F}_i$ If $V \subset U$ note that the restriction $(s_i) \mapsto (s_{i|_V})$ is well defined because : $s_{j|_{U_{ij}\cap V}} = \varphi_{ij}(s_{i|_{U_{ij}}})_{|V} = \varphi_{ij}(s_{i|_{U_{ij}\cap V}})$, using the fact that φ_{ij} is a morphism of sheaves.

- We show that $\mathcal{F}$ is indeed a sheaf. Let $V = \cup_\alpha V_\alpha$ an open cover. Let $((s_i^\alpha)_i)_\alpha$ be a collection of elements lying in $\mathcal{F}(V_\alpha)$, such that we have $s_i^\alpha{}_{|V_\alpha} = s_i^\beta{}_{|V_\beta}$ for any α, β. Using the sheaf property on the product sheaf (which follows directly from the sheaf property of each factor), we get a *unique* element $(s_i) \in \prod_i \iota_{i*}\mathcal{F}_i(V)$ lifting the collection. We show that this unique element lies in fact in $\mathcal{F}(V)$. We need to show that for any i,j we have $s_{j|_{U_{ij}}} = \varphi_{ij}(s_{i|_{U_{ij}}})$. But when we restrict both sides of the desired equality on V_α, the equality holds because $(s_i^\alpha)_i$ lies in $\mathcal{F}(V_\alpha)$. So using again the uniqueness in the sheaf property of the product sheaf, we get what we want.
- To show that $\mathcal{F}$ is unique up to isomorphism in $\mathrm{Sh}(X)$ we spell out an universal property that it verifies. We write $(\mathcal{F} \xrightarrow{p_i} \iota_{i*}\mathcal{F}_i)_i$ the collection of sheaf morphisms induced by the projections from the product. We claim that $\mathcal{F}$ satisfies the following universal property :

  For all $\mathcal{G} \in \mathrm{Sh}(X)$ and collections $(\mathcal{G} \xrightarrow{f_i} \iota_{i*}\mathcal{F}_i)_i$ of sheaf morphisms such that :
  for all U open and $\forall t \in \mathcal{G}(U)$, we have for all i,j :
  $f_j(t)_{|U ij} = \varphi_{ij}(f_i(t)_{|U ij})$
  there is a unique sheaf morphism $\mathcal{G} \xrightarrow{f} \mathcal{F}$ such that for all i,
  $p_i f = f_i$.

  This is indeed the case : if we take a collection $(\mathcal{G} \xrightarrow{f_i} \iota_{i*}\mathcal{F}_i)_i$, we get a map f from $\mathcal{G}$ to the product $\prod_i \iota_{i*}\mathcal{F}_i$ by the universal property

[1] One should question the coherence of this definition : let $(s_i) \in \mathcal{F}(U)$. Then

$$s_{j|_{U_{ij}}} = \varphi_{ij}(s_{i|_{U_{ij}}}) = \varphi_{ij}(\varphi_{ji}(s_{j|_{U_{ij}}})) = s_{j|_{U_{ij}}} \tag{1}$$

using the property for (i,j) and (j,i) and $\mathrm{id}_{\mathcal{F}_j} = \varphi_{ij} \circ \varphi_{ji}$. So (1) highlight why the fact that φ_{ij} and φ_{ji} are inverses to each other is important in this gluing process.

of the product in $\mathrm{Sh}(X)$. But the condition $f_j(t)_{|Uij} = \varphi_{ij}(f_i(t)_{|Uij})$ for all i, j says exactly that in fact f factors into $\mathcal{F}$.

- Now we show that $\varphi_k : \mathcal{F}_{U_k} \to \mathcal{F}_k$ induced by the projection is an isomorphism of sheaves for all k. To show surjectivity we will use crucially the cocycle condition.
  (1) Surjectivity. Let $V \subset U_k$ open. Let $s_k \in \mathcal{F}_k(V)$. We want to construct an element $(s_i) \in \mathcal{F}(V) \subset \prod_i \mathcal{F}_i(V \cap U_i)$ such that its k-th component is s_k.
  For each i, we define s_i using the cover $V \cap U_i = \cup_j V \cap U_{ij}$, and the collection $(\varphi_{ki}(s_{k|Uij}))$ of elements in $\mathcal{F}_i(V \cap U_{ij})$. It verifies the intersection property because φ_{ij} is a morphism of sheaves. So s_i is defined by $s_{i|U_{ij}} = \varphi_{ki}(s_{k|Uij})$. Note that if $i = k$ the element defined in this way is s_k, because φ_{kk} is the identity.
  Now we claim that the collection (s_i) that we just defined is indeed in $\mathcal{F}(V)$. To show this, we need to show that for any i, j, we have : $s_{j|U_{ij}} = \varphi_{ij}(s_{i|U_{ij}})$. But :

$$s_{j|U_{ij}} = \varphi_{kj}(s_{k|Uij}) = \varphi_{ij} \circ \varphi_{ki}(s_{k|Uij}) = \varphi_{ij}(\varphi_{ki}(s_{k|Uij})) = \varphi_{ij}(s_{i|U_{ij}})$$

  Using the definition of s_i's and the cocycle condition.
  (2) Injectivity. Let (s_i) and (s'_i) be two elements in $\mathcal{F}(V)$ such that their k-th component is $s_k = s'_k$. Now one gets for any i :

$$s_i = \varphi_{ik}(s_k) = \varphi_{ik}(s'_k) = s'_i$$

  thus proving the injectivity.

Remark. One can interpret the result of the previous exercise as sying that the presheaf with values in categories

$$\mathrm{Sh} \colon \mathrm{Ouv}(X)^{\mathrm{op}} \to \mathrm{Cat}$$

is a sheaf in a suitable sense.

Exercise 5. Let R be a ring. Let (a_i) be a collection of elements such that

$$\mathrm{Spec}(R) = \bigcup_i D(a_i).$$

This means that $1 \in (a_i)$. Therefore there exists $a_1, \ldots, a_n$ and $b_1, \ldots, b_n \in R$ such that

$$1 = \sum_{j=1}^{n} b_j a_j$$

for some n. Therefore

$$\mathrm{Spec}(R) = \bigcup_{j=1}^{n} D(a_j).$$

Exercise 7. *Stalks, morphisms and cotangent spaces*
This exercise was a previous hand in exercise, and solutions are credited to past students of the course.
(2)(Alissa) Let R be an integral domain. Consider $\phi : R[x, y] \to R[x, y]$ a ring homomorphism such that $x \mapsto xy$ and $y \mapsto y$. Consider now the map

$f : \mathrm{Spec}(R[x,y]) \to \mathrm{Spec}(R[x,y])$ induced by the map ϕ. We show that for every $\lambda \in R$ we have that $f((x-\lambda, y)) = \phi^{-1}(x-\lambda, y) = (x,y)$. To prove this point, consider the following commutative diagram

$$\begin{tikzcd} R[x,y] \arrow[rr, "\phi"] \arrow[dr, "\mathrm{ev}_{(0,0)}"'] & & R[x,y] \arrow[dl, "\mathrm{ev}_{(\lambda,0)}"] \\ & R & \end{tikzcd}$$

We have that $\mathrm{ev}_{(\lambda,0)} \circ \phi = \mathrm{ev}_{(0,0)}$. Hence we have the following series of equalities

$$\begin{aligned}(x,y) &= \ker(\mathrm{ev}_{(0,0)}) = \mathrm{ev}_{(0,0)}^{-1}(0) = (\mathrm{ev}_{(\lambda,0)} \circ \phi)^{-1}(0) \\ &= \phi^{-1}(\mathrm{ev}_{(\lambda,0)}^{-1}(0)) = \phi^{-1}(x-\lambda, y) = f(x-\lambda, y)\end{aligned}$$

This proves our point.

(3)(Maxence) Now let $R = k$ be a field. We have a local homomorphism of local rings

$$f^\sharp_{(x,y)} : \mathcal{O}_{\mathrm{Spec}(k[x,y]),(x,y)} \to \mathcal{O}_{\mathrm{Spec}(k[x,y]),(x-\lambda,y)}.$$

But we know that $\mathcal{O}_{\mathrm{Spec}(k[x,y]),\mathfrak{p}} \cong k[x,y]_\mathfrak{p}$ for any $\mathfrak{p} \in \mathrm{Spec}(k[x,y])$. Thus we can see $f^\sharp_{(x,y)}$ as a local homomorphism of local rings $k[x,y]_{(x,y)} \to k[x,y]_{(x-\lambda,y)}$. Set $\mathfrak{m}_{(0,0)}$ and $\mathfrak{m}_{(\lambda,0)}$ to be the maximal ideal of respectively $k[x,y]_{(x,y)}$ and $k[x,y]_{(x-\lambda,y)}$.

We want to understand the k-linear map $\mathfrak{m}_{(0,0)}/\mathfrak{m}^2_{(0,0)} \to \mathfrak{m}_{(\lambda,0)}/\mathfrak{m}^2_{(\lambda,0)}$. For any maximal ideal $\mathfrak{m}$ of $k[x,y]$, we have the following isomorphism of k-vector spaces ($k[x,y]/\mathfrak{m}$-vector spaces) :

$$\mathfrak{m}_\mathfrak{m}/\mathfrak{m}^2_\mathfrak{m} \cong \mathfrak{m}/\mathfrak{m}^2$$

where $\mathfrak{m}_\mathfrak{m}$ is the maximal ideal of $k[x,y]_\mathfrak{m}$.

Furthermore, it is easy to see that $\{\overline{x}, \overline{y}\}$ is a k-basis of $(x,y)/(x^2, xy, y^2)$ and $\{\overline{x-\lambda}, \overline{y}\}$ is k-basis of $(x-\lambda, y)/((x-\lambda)^2, (x-\lambda)y, y^2)$ since they are k-linear independent elements in their respective quotient. Since the induced linear map is just defined by applying φ, we get that $\varphi(\overline{x}) = \overline{xy} = \lambda\overline{y}$ and $\varphi(\overline{y}) = \overline{y}$ by definition of elements in the quotient $(x-\lambda, y)/((x-\lambda)^2, (x-\lambda)y, y^2)$. That is, by taking bases as above, the linear map that we are looking for can be describe as the following matrix

$$\begin{pmatrix} 0 & 0 \\ \lambda & 1 \end{pmatrix}.$$